# New Nevanlinna matrices for orthogonal polynomials related to cubic birth and death processes

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# New Nevanlinna matrices for orthogonal polynomials related to cubic birth and death processes

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**Abstract**

The orthogonal polynomials with recurrence relation

$$(\lambda_n + \mu_n - z)\, F_n(z) = \mu_{n+1}\, F_{n+1}(z) + \lambda_{n-1}\, F_{n-1}(z)$$

and the three kinds of cubic transition rates

$$\begin{cases} \lambda_n = (3n+1)^2(3n+2), & \mu_n = (3n-1)(3n)^2, \\ \lambda_n = (3n+2)^2(3n+3), & \mu_n = 3n(3n+1)^2, \\ \lambda_n = (3n+1)(3n+2)^2, & \mu_n = (3n)^2(3n+1), \end{cases}$$

correspond to indeterminate Stieltjes moment problems. It follows that the polynomials $F_n(z)$ have infinitely many orthogonality measures, whose Stieltjes transform is obtained from their Nevanlinna matrix, a $2 \times 2$ matrix of entire functions. We present the full Nevanlinna matrix for these three classes of polynomials and we discuss its growthat infinity and the asymptotic behaviour of the spectra of the Nevanlinna extremal measures.

# 1 Birth and death processes and indeterminate moment problems

Birth and death processes are special stationary Markov processes whose state space is the non-negative integers, representing for instance some population. We are interested in the time evolution of a such population, described by the transition probabilities $\mathcal{P}_{m,n}(t)$ which is the probability that the population goes from the state $m$ at time $t = 0$ to the state $n$ at time $t > 0$. This evolution is supposed to be governed on a small time scale by

$$\begin{aligned}
&\mathcal{P}_{n,n+1}(t) = \lambda_n\, t + o(t), \\
&\mathcal{P}_{n,n}(t) = 1 - (\lambda_n + \mu_n)t + o(t), \qquad t \to 0. \\
&\mathcal{P}_{n,n-1}(t) = \mu_n\, t + o(t),
\end{aligned}$$

For applications the most important problem is to find $\mathcal{P}_{m,n}(t)$ for given rates $\lambda_n$ and $\mu_n$, with suitable extra constraints to be described later on.

From the previous setting one can prove that the transition probabilities have to be a solution of the forward Kolmogorov equations

$$\frac{d}{dt}\mathcal{P}_{m,n} = \lambda_{n-1}\, \mathcal{P}_{m,n-1} + \mu_{n+1}\, \mathcal{P}_{m,n+1} - (\lambda_n + \mu_n)\, \mathcal{P}_{m,n}. \tag{1}$$

The $\mathcal{P}_{m,n}(t)$ are assumed to be continuous for small time scales with

$$\lim_{t \to 0}\ \mathcal{P}_{m,n}(t) = \delta_{m,n}. \tag{2}$$

A representation theorem for $\mathcal{P}_{m,n}(t)$ proved by Karlin and MacGregor in [7] links birth and death processes and orthogonal polynomials theory. Let us define the polynomials $F_n(x)$ by the three-terms recurrence relation

$$(\lambda_n + \mu_n - x)F_n(x) = \mu_{n+1}F_{n+1}(x) + \lambda_{n-1}F_{n-1}(x), \qquad n \geq 0, \tag{3}$$

with the boundary conditions

$$F_{-1}(x) = 0, \qquad F_0(x) = 1,$$

and the useful quantities

$$\pi_0 = 1, \qquad \pi_n = \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2 \cdots \mu_n}, \qquad n \geq 1.$$

If the positivity conditions

$$\lambda_n > 0, \qquad n \geq 0, \qquad \text{and} \qquad \mu_0 = 0, \qquad \mu_n > 0, \qquad n \geq 1 \tag{4}$$

are fulfilled, then there is a positive measure $\psi$ for which

$$\mathcal{P}_{m,n}(t, \psi) = \frac{1}{\pi_m} \int_0^\infty e^{-xt} F_m(x) F_n(x)\, d\psi(x). \tag{5}$$

Then the boundary condition (2) is nothing but the orthogonality relation

$$\frac{1}{\pi_m}\int_{\mathrm{supp}\psi} F_m(x)F_n(x)\,d\psi(x) = \delta_{m,n}.$$

Such a measure has well-defined moments

$$c_n = \int_{\mathrm{supp}\psi} x^n\,d\psi(x), \qquad\qquad n = 0,\,1,\ldots.$$

If $\mathrm{supp}(\psi) \subseteq \mathbb{R}$ this is a Hamburger moment problem and if $\mathrm{supp}(\psi) \subseteq [0,\infty)$ this is a Stieltjes moment problem. In the event that the measure $\psi$ is not unique we speak of indeterminate Hamburger (or indeterminate Stieltjes) moment problems, indet H or indet S for short. Stieltjes (see [1]) obtained the necessary and sufficient conditions for a moment problem to be indet S

$$\sum_{n=0}^{\infty}\pi_n < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty}\frac{1}{\lambda_n\pi_n} < \infty.$$

These conditions imply that it is also indet H. They will be supposed to hold in all what follows.

## 1.1 Orthogonality measures for indeterminate moment problems

The description of all the orthogonality measures for an indet H problem was solved a long time ago by Nevanlinna and M. Riesz, with a very good account in [1]. For the reader's convenience let us give a minimal account of this theory. More information will be found in [3]. One first defines the Nevanlinna matrix $\mathcal{N}(z)$ as

$$\begin{pmatrix} A(z) & C(z) \\ B(z) & D(z) \end{pmatrix}, \qquad A(z)\,D(z) - B(z)\,C(z) = 1, \qquad \forall z\in\mathbb{C},$$

where the four entire functions are given by

$$A(z) = z\sum_{n=0}^{\infty}\frac{\widetilde{F}_n(z)}{\mu_1}\sum_{k=1}^{n+1}\frac{1}{\mu_k\pi_k}, \qquad\qquad B(z) = -1 - z\sum_{n=0}^{\infty}F_n(z)\sum_{k=1}^{n}\frac{1}{\mu_k\pi_k},$$

$$C(z) = 1 - z\sum_{n=0}^{\infty}\frac{\widetilde{F}_n(z)}{\mu_1}, \qquad\qquad D(z) = z\sum_{n=0}^{\infty}F_n(z),$$

and the polynomials $\widetilde{F}_n(x)$ are the solution of the recurrence relation (3) with $\lambda_n$and$\mu_n$ replaced by the shift rates $\widetilde{\lambda}_n = \lambda_{n+1}$ and $\widetilde{\mu}_n = \mu_{n+1}$. Once the Nevanlinna matrix is known all of the orthogonality measures $\psi_\phi$ have for Stieltjes transform

$$\int\frac{d\psi_\phi(x)}{z-x} = \frac{A(z)\phi(z) - C(z)}{B(z)\phi(z) - D(z)}, \qquad z\in\mathbb{C}\backslash\mathbb{R},$$

where the Pick function $\phi(z)$ has the general structure

$$\phi(z) = t + s\,z + \int\frac{xz+1}{x-z}\,d\nu(x), \qquad t\in\mathbb{R}, \quad s\geq 0,$$

where $\nu$ is a finite positive measure supported on $\mathbb{R}$.

The Nevanlinna extremal (for short N-extremal) measures $\psi_t$ correspond to the Pick function $\phi(z) = t$. These measures are fully discrete and for them and only for them are the polynomials $\{F_n,\ n = 0, 1, \ldots\}$ dense in $L^2(\mathbb{R}, d\psi_t)$.

The number

$$-\frac{1}{\alpha} = \sum_{n=1}^{\infty} \frac{1}{\mu_n \pi_n},$$

is quite important since the positively supported N-extremal measures (hence corresponding to S moment problems) are given by $t \in [\alpha, 0]$.

Among all of these positively supported measures, there is a unique one $\psi_0(x)$, which has a mass at $x = 0$ and corresponds to the parameter $t = 0$. In [7] it is called the "minimal" measure since it gives rise to "minimal" transition probabilities as explained page 526 of this reference.

The measure corresponding to $t = \alpha$ has been shown by Pedersen [8] to correspond to the Friedrichs extension of the Jacobi matrix and could be called the Friedrichs measure. Notice also that necessary and sufficient conditions on the Pick function are known which ensure that the corresponding measure is positively supported [9].

## 1.2 More restrictions on the measures

Let us now present further restrictions on the transition probabilities and see what consequences and restrictions we get on the measures $\psi$. We thought it could be useful for the reader to have a global view of these beautiful results, the proofs of which were given in [7].

1. The first obvious restriction is that the transition probabilities are *positive*: as soon as $\psi$ is a solution of the H moment problem then one has

$$\mathcal{P}_{m,n}(t, \psi) > 0 \qquad \forall m \geq 0, \ \ \forall n \geq 0, \qquad \forall t > 0.$$

2. The semi-group property

$$\sum_{l=0}^{\infty} \mathcal{P}_{m,l}(s, \psi)\, \mathcal{P}_{l,n}(t, \psi) = \mathcal{P}_{m,n}(s + t, \psi), \qquad s \geq 0, \quad t \geq 0$$

holds iff $\psi$ is N-extremal.

3. Before looking at the Markov property one could impose the weaker restrictions

$$0 < \sum_{n=0}^{\infty} \mathcal{P}_{m,n}(t, \psi) < 1 \qquad t > 0.$$

These will hold if $\psi$ is a solution of the S moment problem.

4. The Markov property [1]

$$\sum_{n=0}^{\infty} \mathcal{P}_{m,n}(t, \psi) = 1 \qquad t > 0$$

[1] In the former terminology a process for which the Markov property holds was denoted as "honest".

holds for

$$\mathcal{P}_{m,n}(t,\psi_0) = \frac{1}{\pi_m}\int_0^\infty e^{-xt}F_m(x)F_n(x)\,d\psi_0(x),$$

where $\psi_0$ is the "minimal" measure of Karlin and MacGregor, as defined in the previous section.

For practical applications the Markov property is of paramount importance, and selects among all possible N-extremal measures a unique one. From this point of view the uniqueness of the solution of the Kolmogorov equation is restored by the Markov property and applies only to $\mathcal{P}_{m,n}(t,\psi_0)$.

If we consider *polynomial* transition rates of the form

$$\lambda_n = (n+b_1)\cdots(n+b_p), \qquad \mu_n = (n+a_1)\cdots(n+a_p)(1-\delta_{n0}), \qquad n\geq 0,$$

we have for large $n$

$$\pi_n = \mathcal{O}(n^{\Delta-p}), \qquad \frac{1}{\lambda_n\pi_n} = \mathcal{O}(n^{-\Delta}), \qquad \Delta = \sum_{i=1}^{p}(b_i-a_i).$$

The moment problem is indet S for $1<\Delta<p-1$, and this requires at least cubic transition rates. In this class only a few cases have been worked out, mainly the quartic example [10] with

$$\lambda_n = (4n+4c+1)(4n+4c+2)^2(4n+4c+3), \qquad \mu_n = (4n+4c-1)(4n+4c)^2(4n+4c+1)(1-\delta_{n0}),$$

where the Nevanlinna matrix is known but leads to no explicit N-extremal measure. In the limiting case $c\to 0$ the "minimal" measure (which enjoys the Markov property) and the Friedrichs measure were obtained explicitly, while for $c=1/4$ some new explicit N-extremal measures are given by Christiansen [5].

# 2 New Nevanlinna matrices

We will describe, in the sequel, some Nevanlinna matrices for some particular choices of cubic $\lambda_n$ and $\mu_n$. To this end we need some background material.

## 2.1 Background material

In order to describe the Nevanlinna matrix we will need a triplet of elementary functions defined by

$$\sigma_l(u) = \sum_{n=0}^{\infty}(-1)^n\,\frac{u^{3n+l}}{(3n+l)!}, \qquad l=0,1,2. \tag{6}$$

It is easy to check the relations

$$\sigma_1' = \sigma_0, \qquad \sigma_2' = \sigma_1, \qquad \sigma_0' = -\sigma_2, \tag{7}$$

with the boundary conditions

$$\sigma_0(0)=1, \qquad \sigma_1(0)=0, \qquad \sigma_2(0)=0.$$

These functions are called trigonometric functions of order 3, since they are three linearly independent solutions of the third order differential equation

$$\sigma_l''' + \sigma_l = 0, \qquad l = 0,\, 1,\, 2.$$

Their explicit form is

$$\begin{aligned}
\sigma_0(u) &= \frac{1}{3}\left(e^{-u} + 2\cos\left(\frac{\sqrt{3}}{2}u\right)e^{u/2}\right),\\
\sigma_1(u) &= \frac{1}{3}\left(-e^{-u} + 2\cos\left(\frac{\sqrt{3}}{2}u - \frac{\pi}{3}\right)e^{u/2}\right),\\
\sigma_2(u) &= \frac{1}{3}\left(e^{-u} - 2\cos\left(\frac{\sqrt{3}}{2}u + \frac{\pi}{3}\right)e^{u/2}\right).
\end{aligned} \tag{8}$$

We will need also the following functions

$$\theta(t) = \int_0^t \frac{du}{(1-u^3)^{2/3}}, \qquad \widehat{\theta}(t) = \theta_0 - \theta(t), \qquad \theta_0 \equiv \theta(1) = \frac{\Gamma^3(1/3)}{2\pi\sqrt{3}}. \tag{9}$$

Notice that $\widehat{\theta}(t)$ is continuous for $t \in [0,1]$ with the bounds $0 \le \widehat{\theta}(t) \le \theta_0$.

## 2.2 The Nevanlinna matrices

In [6] the Nevanlinna matrix of the polynomials $F_n(x)$, with the transition rates

$$\lambda_n = (3n+3c+1)^2(3n+3c+2), \qquad \mu_n = (3n+3c-1)(3n+3c)^2(1-\delta_{n0}), \quad c \ge 0$$

has been obtained through the computation of some generating functions. From the previous section we can check that it is an indeterminate Stieltjes moment problem since we have $\Delta = 5/3$. Important simplifications in the structure of the Nevanlinna matrix occur for the special cases $c = 0$ and $c = 1/3$. Similar results were also obtained for the recurrence coefficients

$$\lambda_n = (3n+3c+1)(3n+3c+2)^2, \qquad \mu_n = (3n+3c)^2(3n+3c+1)(1-\delta_{n0}), \quad c \ge 0.$$

In this case simplifications do occur only for $c = 0$, the other possibility $c = -1/3$ is ruled out by positivity since it leads to $\lambda_0 = 0$. These simplified cases will be studied in detail since they allow further insights into the N-extremal spectra and the growth at infinity.

Let us observe that the computational techniques used in the previous reference allow to compute $\widetilde{A}$ and $\widetilde{B}$ instead of $A$ and $B$. We have the relations

$$\widetilde{A}(z) = A(z) - \frac{C(z)}{\alpha}, \qquad \widetilde{B}(z) = B(z) - \frac{D(z)}{\alpha}, \qquad -\frac{1}{\alpha} = \sum_{n=1}^{\infty}\frac{1}{\mu_n\pi_n}.$$

We use this transformation in order in the following to give simplified formulas (for example (10-12)).The sum defining $\alpha$ converges since the moment problem is indet S.

In the first case with $c = 0$ the recurrence coefficients are

$$\lambda_n = (3n+1)^2(3n+2), \qquad \mu_n = (3n-1)(3n)^2, \quad n \ge 0$$

and the corresponding Nevanlinna matrix $\mathcal{N}_0(z)$ is given by

$$\left\{\begin{array}{ll} \widetilde{A}_0(z) = \displaystyle\int_0^1 \frac{\sigma_2(\rho\widehat{\theta}(u))}{\rho^2}\,(1-u^3)^{-1/3}\,du, & \widetilde{B}_0(z) = -\,\sigma_0(\rho\theta_0), \\ C_0(z) = \displaystyle\frac{3\sqrt{3}}{2\pi}\int_0^1 \sigma_0(\rho\widehat{\theta}(u))\,(1-u^3)^{-1/3}\,du, & D_0(z) = \displaystyle\frac{3\sqrt{3}}{2\pi}\,\rho^2\,\sigma_1(\rho\theta_0), \end{array}\right. \tag{10}$$

with $\rho = z^{1/3}$. Our results for $\widetilde{B}_0(z)$ are in agreement with [11], but correct the mistaken form of $D_0(z)$ given in this reference.

In the second case with $c = 1/3$ the recurrence coefficients are

$$\lambda_n = (3n+2)^2(3n+3), \qquad \mu_n = 3n(3n+1)^2, \quad n \geq 0,$$

and the corresponding Nevanlinna matrix $\mathcal{N}_1(z)$ is given by

$$\left\{\begin{array}{ll} \widetilde{A}_1(z) = \displaystyle\int_0^1 \frac{\sigma_2(\rho\widehat{\theta}(u))}{\rho^2}\,u\,(1-u^3)^{-1/3}\,du, & \widetilde{B}_1(z) = -\displaystyle\frac{\sigma_1(\rho\theta_0)}{\rho}, \\ C_1(z) = \displaystyle\frac{3}{B(\frac{2}{3},\frac{2}{3})}\int_0^1 \sigma_0(\rho\widehat{\theta}(u))\,u\,(1-u^3)^{-1/3}\,du, & D_1(z) = \displaystyle\frac{3}{B(\frac{2}{3},\frac{2}{3})}\,\rho\,\sigma_2(\rho\theta_0). \end{array}\right. \tag{11}$$

In the third case with again $c = 0$ the recurrence coefficients are

$$\lambda_n = (3n+1)(3n+2)^2, \qquad \mu_n = (3n)^2(3n+1), \quad n \geq 0,$$

and the corresponding Nevanlinna matrix $\mathcal{N}_2(z)$ is given by

$$\left\{\begin{array}{ll} \widetilde{A}_2(z) = \displaystyle\int_0^1 \frac{\sigma_2(\rho\widehat{\theta}(u))}{\rho^2}\,du, & \widetilde{B}_2(z) = -\sigma_0(\rho\theta_0), \\ C_2(z) = \displaystyle\frac{3\sqrt{3}}{2\pi}\int_0^1 \frac{\sigma_1(\rho\widehat{\theta}(u))}{\rho}\,du, & D_2(z) = \displaystyle\frac{3\sqrt{3}}{2\pi}\,\rho\,\sigma_2(\rho\theta_0). \end{array}\right. \tag{12}$$

All the functions are seen to be entire upon use of relation (6) and the couples $\widetilde{B}$ and $D$ exhaust all possible couples made out of the four entire functions $\rho^{-l}\sigma_l(\rho\theta_0)$, $\rho^{2-l}\sigma_{l+1}(\rho\theta_0)$ with $l = 0, 1$.

# 3 Some applications

As an application of the previous results, we will discuss two significant aspects of the Nevanlinna matrices: their growth at infinity and the asymptotics of the N-extremal spectra.

## 3.1 Growth at infinity of the Nevanlinna matrix

The growth at infinity of the entire functions is described mainly in terms of the order $\eta$, type $\sigma$, and Phragmén-Lindelöf indicator $h(\theta)$. Let us begin with

**Proposition 1** *The Nevanlinna matrices $\mathcal{N}_0$, $\mathcal{N}_1$ and $\mathcal{N}_2$ have the same order $\eta$ and type $\sigma$ given by*

$$\eta = \frac{1}{3}, \qquad \sigma = \theta_0 = \int_0^1 \frac{du}{(1-u^3)^{2/3}}$$

**Proof :**
As shown in [2] the order and type of all the elements of a given Nevanlinna matrix are one and the same. Hence it will be sufficient to determine them for the simplest elements i.e. for $D_0$ and $D_1$ since $D_2$ is proportional to $D_1$. Using the expansions (6) we can write

$$D_l(z) = \sum_{n=1}^{\infty} (-1)^{n-1}\, \xi_n^{(l)}\, z^n,$$

with

$$\xi_n^{(0)} = \frac{3\sqrt{3}}{2\pi}\,\frac{\theta_0^{3n-2}}{(3n-2)!}, \qquad \xi_n^{(1)} = \frac{3}{B(\frac{2}{3},\frac{2}{3})}\,\frac{\theta_0^{3n-1}}{(3n-1)!}, \tag{13}$$

and then we compute the order $\eta$ and type $\sigma$ from [4]

$$\eta = \overline{\lim_{n\to\infty}}\,\frac{n\ln n}{|ln|\xi_n||}, \qquad \sigma = \frac{1}{e\rho}\,\overline{\lim_{n\to\infty}}\, n\,|\xi_n|^{\rho/n}. \tag{14}$$

Using Stirling's formula one obtains the results stated above. ▮

Let us turn ourselves to the Phragmén-Lindelöf indicator, which is defined by

$$h_f(\theta) = \overline{\lim_{r\to\infty}}\,\frac{\ln|f(re^{i\theta})|}{r^\rho}. \tag{15}$$

We will prove that the Nevanlinna matrices $\mathcal{N}_0$, $\mathcal{N}_1$ and $\mathcal{N}_2$ have the same Phragmén-Lindelöf indicator. To this end we first prove

**Proposition 2** *The two entire functions $\widetilde{B}_0$ and $\widetilde{B}_1$ have the same Phragmén-Lindelöf indicator: $h(\theta) = \theta_0\cos(\frac{\theta-\pi}{3})$, $\theta \in [0, 2\pi[$.*

**Proof :**
We will use the relation

$$\sigma_0(\zeta) = \frac{1}{3}\left(e^{-\zeta} + e^{j\zeta} + e^{\bar{j}\zeta}\right), \qquad j = e^{i\frac{\pi}{3}}. \tag{16}$$

For $z = re^{i\theta}$ we have $\zeta = \rho\,\theta_0 = u\,e^{i\theta/3}$ with $u = r^{1/3}\,\theta_0$

Elementary algebra gives

$$|e^{-\zeta}| = e^{-u\cos(\frac{\theta}{3})}, \qquad |e^{j\zeta}| = e^{u\cos(\frac{\theta+\pi}{3})}, \qquad |e^{\bar{j}\zeta}| = e^{u\cos(\frac{\theta-\pi}{3})}.$$

For $\theta \in [0, 2\pi[$ we have the inequalities

$$\cos(\frac{\theta-\pi}{3}) > -\cos(\frac{\theta}{3}), \qquad \cos(\frac{\theta-\pi}{3}) \geq \cos(\frac{\theta+\pi}{3}), \tag{17}$$

hence we get

$$3|\sigma_0(ue^{i\theta/3})| = \mathcal{O}(\gamma\; e^{u\cos(\frac{\theta-\pi}{3})}), \qquad u\to\infty,$$

with $\gamma = 1,\ \theta \in ]0, 2\pi[$ and $\gamma = 2$, for $\theta = 0$. This allows to compute the indicator

$$h(\theta) = \theta_0\,\cos(\frac{\theta - \pi}{3}), \qquad \theta \in\ [0, 2\pi[. \tag{18}$$

The argument is similar for $\sigma_1$ which is given by

$$\sigma_1(\zeta) = \frac{1}{3}\left(-e^{-\zeta} + \overline{j}\,e^{j\zeta} + j\,e^{\overline{j}\zeta}\right).$$

Notice that we have also to use the obvious observation that $f(z)$ and $z^\nu f(z)$ with $\nu \in \mathbb{R}$ have the same Phragmén-Lindelöf indicator. ▮

As a check let us observe that the maximum value of the indicator agrees with the type $\sigma = \theta_0$ determined above.

We conclude to

**Proposition 3** *The Nevanlinna matrices $\mathcal{N}_l$, with $l = 0, 1, 2$ have the same Phragmén-Lindelöf indicator (18).*

**Proof :**
Using the results in [2], if $0 < \eta < \infty$, $0 < \sigma < \infty$ then $\widetilde{A}_l,\ \widetilde{B}_l,\ C_l$ and $D_l$ have the same Phragmén-Lindelöf indicator for $l = 0, 1, 2$. ▮

## 3.2 Asymptotic behaviour of the N-extremal spectra

The N-extremal mass point, for a given Nevanlinna matrix, are given by the roots of

$$t\,B(x) - D(x) = 0, \qquad t \in \mathbb{R} \cup \{\infty\},$$

which is easily transformed into

$$t^\sharp\,\widetilde{B}(x) + D(x) = 0, \qquad t^\sharp = \frac{\alpha t}{t - \alpha}. \tag{19}$$

The positively supported measures correspond now to $0 \le t^\sharp \le \infty$. The "minimal" measure ( $t = 0$ ) is mapped into $t^\sharp = 0$ and the Friedrich's measure ( $t = \alpha$ ) is mapped into $t^\sharp = \infty$. For negative $t^\sharp$, as observed in [3] a single mass, negatively supported, appears. Let us prove:

**Proposition 4** *The asymptotic behaviour of the N-extremal mass point is independent of $t^\sharp$, is the same for $\mathcal{N}_0, \mathcal{N}_1$ and $\mathcal{N}_2$ and is given by*

$$x_n = \left(\frac{2\pi n}{\sqrt{3}\theta_0}\right)^3 + o(n^3), \qquad n \to \infty. \tag{20}$$

**Proof :**
Let us begin with $\mathcal{N}_0$. The explicit form of the relation (19) is

$$\widetilde{t}\,\sigma_0(u) = u^2\,\sigma_1(u), \qquad u = x^{1/3}\theta_0,$$

where $\widetilde{t}$ is proportional to $t^\sharp$. For $\widetilde{t}=\infty$ this relation becomes $\sigma_0(u)=0$ which leads to

$$\cos\left(\frac{\sqrt{3}}{2}u\right)=-\frac{1}{2}\,e^{-3u/2}\qquad\Rightarrow\qquad u_n=\frac{2\pi}{\sqrt{3}}\,n+\mathcal{O}(1),\qquad\quad n\to\infty\ .$$

For $0\le\widetilde{t}<\infty$ the eigenvalue relation can be written

$$\cos\left(\frac{\sqrt{3}}{2}u-\frac{\pi}{3}\right)=\frac{1}{2}\,e^{-3u/2}+\frac{\widetilde{t}}{u^2}\left(\frac{1}{2}\,e^{-3u/2}+\cos\left(\frac{\sqrt{3}}{2}u\right)\,e^{u/2}\right),$$

which gives

$$\frac{\sqrt{3}}{2}u_n=(n+1/2)\pi+\mathcal{O}(1)=n\,\pi+\mathcal{O}(1),$$

and for large $n$ this is the same behaviour as for $\widetilde{t}=\infty$.

Let us proceed with $\mathcal{N}_1$. The explicit form of the relation (19) is now

$$\widetilde{t}\,\sigma_1(u)=u^2\,\sigma_2(u).$$

For $\widetilde{t}=\infty$ this relation becomes $\sigma_1(u)=0$ and leads to

$$\cos\left(\frac{\sqrt{3}}{2}u-\frac{\pi}{3}\right)=\frac{1}{2}\,e^{-3u/2},$$

which gives the required asymptotics. For $0\le\widetilde{t}<\infty$, the eigenvalue equation becomes

$$\cos\left(\frac{\sqrt{3}}{2}\,u+\frac{\pi}{3}\right)=\frac{1}{2}\,e^{-3u/2}+\frac{\widetilde{t}}{u^2}\left(\frac{1}{2}\,e^{-3u/2}-\cos\left(\frac{\sqrt{3}}{2}u-\frac{\pi}{3}\right)\right),$$

and this leads to

$$\frac{\sqrt{3}}{2}u_n=-\frac{\pi}{3}+(n+1/2)\pi+\mathcal{O}(1)=n\pi+\mathcal{O}(1).$$

Let us conclude with $\mathcal{N}_2$. The explicit form of the relation (19) is now

$$\widetilde{t}\,\sigma_0(u)=u\,\sigma_2(u).$$

For $\widetilde{t}=\infty$ this relation becomes $\sigma_0(u)=0$ showing that the "minimal" spectrum of $\mathcal{N}_0$ does coincide with the "minimal" spectrum of $\mathcal{N}_2$, and has therefore the same asymptotics. For $0\le\widetilde{t}<\infty$, the eigenvalue equation becomes

$$\cos\left(\frac{\sqrt{3}}{2}\,u+\frac{\pi}{3}\right)=\frac{1}{2}\,e^{-3u/2}-\frac{\widetilde{t}}{u}\left(\frac{1}{2}\,e^{-3u/2}+\cos\left(\frac{\sqrt{3}}{2}u\right)\right),$$

and this leads to

$$\frac{\sqrt{3}}{2}u_n\sim-\frac{\pi}{3}+(n+1/2)\pi=n\pi+\mathcal{O}(1),$$

concluding the proof. ▮

Let us point out that the law $x_n=\mathcal{O}(n^3)$ is no surprise, since this implies that the order of the Nevanlinna matrices is always 1/3 and this we knew from Proposition 1. The significant result is the determination of the coefficient in front of $n^3$.

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